Problem 01: Bounds

Given an integer function f and a non-empty interval $[m, n]$, compute the minimal and maximal values and positions of f in the interval.

$$\begin{array}{rcccccccccccc} A & = & \mathbb{Z} & \times & \mathbb{Z} & \times & \mathbb{Z} & \times & \mathbb{Z} & \times & \mathbb{Z} & \times & \mathbb{Z} \\ & & m & & n & & u & & min & & v & & max & & i \\ B & = & \mathbb{Z} & \times & \mathbb{Z} & & & & & & & & & & \\ & & m' & & n' & & & & & & & & & & \end{array}$$

$$Q = (m' = m) \wedge (n' = n) \wedge (m \leq n)$$

$$R = Q \wedge u, v \in [m, n] \wedge \forall k \in [m, n] : min \leq f(k) \leq max \wedge f(u) = min \wedge f(v) = max$$

Solution

We can easily solve this with the following loop:

$$P = Q \wedge i \in [m, n] \wedge u, v \in [m, i] \wedge \forall k \in [m, i] : min \leq f(k) \leq max \wedge f(u) = min \wedge f(v) = max$$

$$\neg\pi = (i = n), \pi = (i \neq n), t = (n - i)$$

Of course, since $Q \not\Rightarrow P$, we'll need an intermediate condition $Q' = Q \wedge (i, u, v = m) \wedge (min, max = f(m))$, which can be easily reached by an appropriate assignment of i, u, and v.

Solving P for $i \leftarrow i + 1$ gives

$$P^{i \leftarrow i+1} = Q \wedge (i + 1) \in [m, n] \wedge u, v \in [m, i + 1] \wedge \forall k \in [m, i + 1] : min \leq f(k) \leq max$$

$$P^{i \leftarrow i+1} \simeq P \wedge min \leq f(i + 1) \leq max$$

Which leads to the following program:

<table>
<tr><td colspan="4">$i, u, v, min, max := m, m, m, f(m), f(m)$</td></tr>
<tr><td rowspan="3"></td><td colspan="3">$i \neq n$</td></tr>
<tr><td>$f(i+1) \leq min$</td><td>$min \leq f(i+1) \leq max$</td><td>$f(i+1) \geq max$</td></tr>
<tr><td>$u, min := i + 1, f(i + 1)$</td><td>$SKIP$</td><td>$v, max := i + 1, f(i + 1)$</td></tr>
<tr><td colspan="4">$i := i + 1$</td></tr>
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